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THESIS

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RADIATION STRESS PHENOMENA

by

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April 1970

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Cnoidal Wave Theory Applied to
Radiation Stress Phenomena

by

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Abstract

A simplified hyperbolic approximation to cnoidal wave theory is applied to generate the radiation stress tensor and used in the equations of motion to obtain solutions for changes in the mean water level outside and inside the surf zone. Comparison between linear theory, cnoidal theory and laboratory results are made. A limiting case and an optimum case for cnoidal theory are discussed in the comparison. Cnoidal theory is shown to give better predictions than linear theory to the data considered.

TABLE OF CONTENTS

I. INTRODUCTION -- 11

II. HISTORICAL SUMMARY OF CNOIDAL THEORY -------------- 13

III. DEVELOPMENT OF HYPERBOLIC WAVE EQUATIONS ---------- 20

IV. RADIATION STRESS PHENOMENA ------------------------ 25

A. DEVELOPMENT OF RADIATION STRESS PHENOMENA ---- 26

1. Development of the Radiation Stress Tensor ----------------------------------- 27

2. Wave Set-down ---------------------------- 31

3. Wave Set-up ------------------------------ 33

V. CNOIDAL WAVE CHARACTERISTICS APPLIED TO RADIATION STRESS EQUATIONS ----------------------------------- 34

A. THE RADIATION STRESS TENSOR ------------------ 34

B. CNOIDAL WAVE SOLUTION FOR SET-DOWN ----------- 37

C. CNOIDAL WAVE SOLUTION FOR SET-UP ------------- 39

VI. APPLICATION OF THEORY TO EXPERIMENTAL DATA -------- 41

VII. RECOMMENDATIONS FOR FUTURE STUDY ------------------ 45

VIII. SUMMARY AND CONCLUSIONS --------------------------- 46

LIST OF REFERENCES ------------------------------------- 48

INITIAL DISTRIBUTION LIST ------------------------------ 50

FORM DD 1473 --- 51

LIST OF FIGURES

1. Cnoidal Wave Profile -------------------------------------18

2. Experimental and Theoretical Wave Profiles ---------18

3. Relationships Between Elliptic Integrals and Elliptic Modulus ---21

4. Hyperbolic Wave Profile and Characteristics --------21

5. Wave Model Coordinate System -----------------------28

6. Plot of Linear and Cnoidal Theories vs. Laboratory Measurements --43

TABLE OF SYMBOLS AND ABBREVIATIONS

C	wave celerity
C_g	wave group velocity
cn	elliptic cosine
D	total water depth
E	Jacobian elliptic integral of the second kind
E_p	potential energy per unit surface area
E_k	kinetic energy per unit surface area
E_t	total energy per unit surface area
F	energy flux
g	acceleration due to gravity
H	wave height
H_o	deep water wave height
h	still water depth
h_t	water depth below the wave trough
K	Jacobian elliptic integral of the first kind
k	elliptic modulus
L	wave length
ℓ_i	complement to angle of incidence
m	wave number
m_o	deep water wave number
$\tilde{M}$	total horizontal momentum
p	pressure
Q	argument of the hyperbolic function
R_i	friction term
S_{ij}	radiation stress tensor

sn	elliptic sine
t	time variable
T_i	force
u_x	X-directed velocity
u_y	Y-directed velocity
$\tilde{U}$	mean steady motion
w	vertical velocity
x,y,z	direction of coordinate axes
α	angle between wave front and bottom contour
β	beach slope angle
δ	distance between still water and wave trough
η	wave profile
η_o	wave profile above still water
$\bar{\eta}$	mean water level
θ	dimensionless parameter
κ	shoaling coefficient
ξ	dimensionless parameter
ϕ	velocity potential
ρ	water density
σ	radial frequency
∇	designation of still water level

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I. INTRODUCTION

The narrow region where the beaches of land masses and the waters of the oceans meet is one of dynamic character. Within the relatively short distance of perhaps a few hundred yards, the waves rapidly release energy which may have been transmitted many hundreds of miles. For the surfer waiting for the perfect wave, the coastal engineer designing a new structure, or the naval commander planning an amphibious operation, the zone of wave energy dissipation against a beach is one of prime importance.

The waves under consideration are those which have been generated by a distant wind system. They have propagated over a relatively long distance and are rather orderly and swell-like in appearance. As they move toward the beach into progressively more shallow water, the waves undergo a transformation. They begin to slow and the wave heights build until the wave can no longer maintain stability. At this point the wave breaks and releases its energy across the surf zone.

Investigators have shown that the unsteady motion of the waves results in a flux of excess momentum, commonly termed a "radiation stress". A change in the radiation stress as a wave train moves into shallow water can cause a suppression of the mean water level outside the surf zone and an elevation of the mean water level inside the surf zone.

Historically, waves of this type most often have been mathematically represented by linear or higher order Stokes'

theories. However, some investigators are of the opinion that cnoidal wave theory, which is a shallow water, finite wave theory, is a more accurate representation of waves in the field; this is because the wave form described by cnoidal theory better resembles actual wave profiles.

This investigation uses cnoidal theory to examine two radiation stress phenomena. A brief history of cnoidal theory is presented to provide some understanding of this class of waves. Next an asymptotic form of cnoidal theory using the "hyperbolic wave" approximation is made and the necessary wave parameters are defined. A brief discussion of the history of the radiation stress follows, with the development of the set-down and set-up expressions concluding the section. The next section developes the application of cnoidal theory to the radiation stress relationships. The following section discusses the application of both linear and cnoidal theory to laboratory data. The two theories are compared to determine their applicability to a particular laboratory experiment and to prototype conditions in general. The final two sections contain recommendations for further investigations and the conclusions reached from this study.

II. HISTORICAL SUMMARY OF CNOIDAL THEORY

This section comprises a brief historical summary of cnoidal wave theory. The origins and major contributions to the theory are included. These are by no means the entire field of contributions and are presented merely as a means of gaining basic familiarity with the theory.

Cnoidal theory was originally developed from an investigation of long wave propagation in a channel by Kortweg and deVries [1]. Their original concern was the solitary wave, a wave of translation, whose wavelength is infinite. The solitary wave form is entirely within the crest and lacks a trough. During their investigation the authors developed a new class of shallow water waves, which are described in terms of Jacobian elliptic functions and integrals. The primary function is the Jacobian elliptic cosine, cn, and from this function came the term cnoidal as an analogy to sinusoidal waves. By varying the Jacobian integrals and functions, one can use cnoidal theory in its asymptotic forms to span the region from Stokes' waves to solitary waves. In general, the cnoidal wave appears as a periodic wave form with shallow troughs and peaked crests (Figure 1).

The development of cnoidal wave theory is in some respects similar to that of Stokes' waves. The assumptions made for the fluid (incompressible, frictionless, homogeneous) are the same. The consideration of the equations of motion, the equations of continuity and appropriate boundary conditions, as well as irrotationality, is made in both theories. However,

cnoidal waves are specifically long waves of finite amplitude and are applicable only in shallow water.

A comparison of the linear long wave solution and first order cnoidal waves is now discussed. Both solutions begin by considering the free surface boundary conditions.

The dynamic free surface boundary condition is written:

$$\frac{\partial \phi}{\partial t} - g\eta - \frac{1}{2}(u^2 + w^2) = 0, \text{ at } Z = \eta \quad . \tag{1}$$

The kinematic free surface boundary condition is:

$$w = \frac{\partial \eta}{\partial t} = \frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x}, \text{ at } Z = \eta \quad . \tag{2}$$

For the linear theory solution and the "zeroth" order cnoidal theory, the assumptions are made that:

$$u^2 \text{ and } w^2 << g\eta$$

and

$$u\frac{\partial u}{\partial x} << \frac{\partial \eta}{\partial t} \quad .$$

Thus for the first approximation, equations (1) and (2) may be reduced to

$$\frac{\partial \phi_o}{\partial t} = g\eta \tag{3}$$

and

$$\frac{\partial \phi_o}{\partial z} + \frac{\partial \eta}{\partial t} = 0 \tag{4}$$

where the subscript represents the first approximation.

The coninuity equation for an irrotational, inviscid and incompressible fluid is represented by Laplace's equation,

$$\nabla^2 \phi = 0 \quad . \tag{5}$$

From equations (3), (4) and (5), Lamb [2] presents the classical long wave equations

$$\frac{\partial^2 u_o}{\partial t^2} = C^2 \frac{\partial^2 u_o}{\partial x^2} \tag{6}$$

and

$$\frac{\partial^2 \eta}{\partial t^2} = C^2 \frac{\partial^2 \eta}{\partial x^2} \tag{7}$$

in which $C^2 = gh$.

A general solution for a progressive wave is now assumed as

$$\frac{\eta}{h} = f\ (x - ct) \tag{8}$$

and

$$\frac{u_o}{c} = f\ (x - ct) \quad . \tag{9}$$

From equations (8) and (9), the solution for the first approximation to the velocity is given as:

$$u_o = C \frac{\eta}{h} \quad . \tag{10}$$

Equation (10) specifies a velocity which is uniform over depth.

When the ratio of $\frac{h}{H}$ and the curvature of the wave profile become significant, the development is no longer satisfactory and a second approximation must be considered. A second approximation is obtained by substituting the results of (10) into equation (1) and proceeding through the same steps as before. As in the first approximation, the vertical

velocity contributions are neglected due to the shallow water considerations. This development results in a differential equation whose solution was originally shown by Kortweg and deVries to be the basis for cnoidal wave theory. This necessarily restricts cnoidal wave theory to shallow water applications.

Keulegan and Patterson [3] used essentially this development in their study of flood waves and other waves of translation moving in an open channel. They developed expressions for the wave profile, wave celerity, and energy for use in examining the deformation of the wave form while undergoing propagation.

Using the perturbation method of expansion, Laitone [4] has developed a second order approximation to cnoidal theory. He notes that an expansion which retains only first order terms gives a solution identical to the original formulation of Kortweg and deVries. From his second order approximation, Laitone notes that pressure variations are no longer merely hydrostatic and that vertical velocity terms may no longer be small enough to ignore.

In a later work, Laitone [5] compared his second approximation of cnoidal theory to third order Stokes' theory and established limiting criteria for the application of these theories to finite amplitude waves. Laitone concluded that the second cnoidal approximation should not be applied to waves whose wavelengths are shorter than five times the water depth. He also established the maximum theoretical

value for the cnoidal shoaling coefficient of $\kappa = .7273$. Laitone concluded that third order Stokes' theory should be limited to cases where the wavelength is less than eight times the water depth.

It should be noted at this point that cnoidal theory is not widely used. This is due to the relative unfamiliarity of the elliptic functions and integrals and to the difficulty in applying them. However, there have been attempts to present cnoidal theory in a more useable form for the engineer. Weigel[6] summarized the theory and compiled tables and graphs of important parameters and functions. Weigel also has made some interesting comparisons of various theoretical and measured wave profiles, which support the supposition that cnoidal theory is more applicable than Stokes' theory in shallow water. (See Figure 2). Masch and Weigel [7] have prepared improved tables of functions and parameters for cnoidal theory, and though this facilitates the use of the theory somewhat, it nevertheless remains relatively cumbersome and difficult to apply.

The mass transport of cnoidal waves was considered by LeMéhauté [8] to treat the run-up of long waves. The development utilizes the wave parameters derived from Laitone's second order work.

A recent application of cnoidal theory has been carried out by Iwagaki [9]. In an attempt to simplify the existing theory to facilitate its use, Iwagaki defines a new class of waves known as "hyperbolic" waves. The elliptic functions are

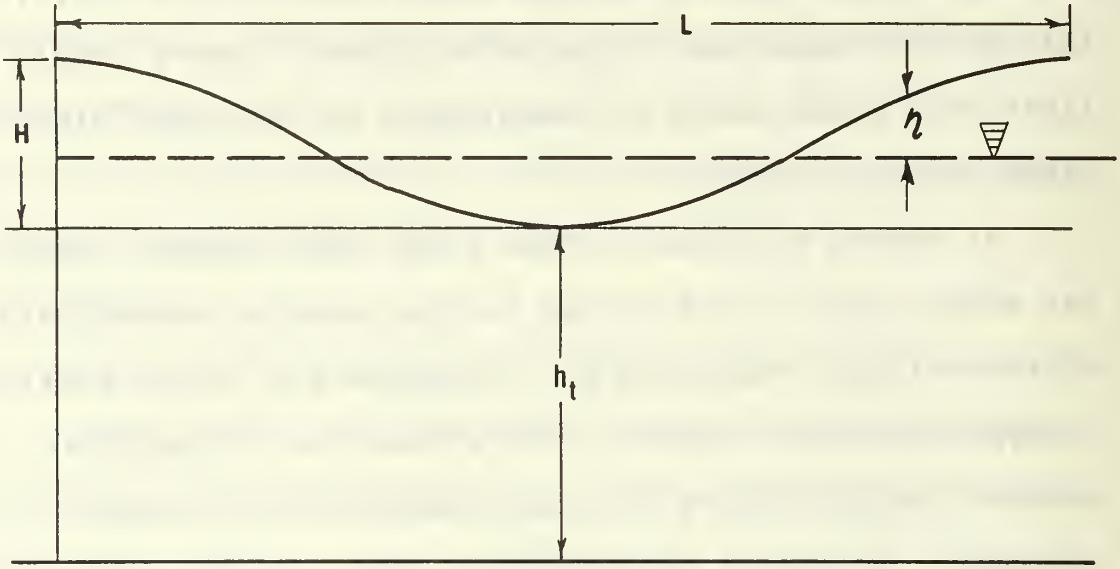

Figure 1

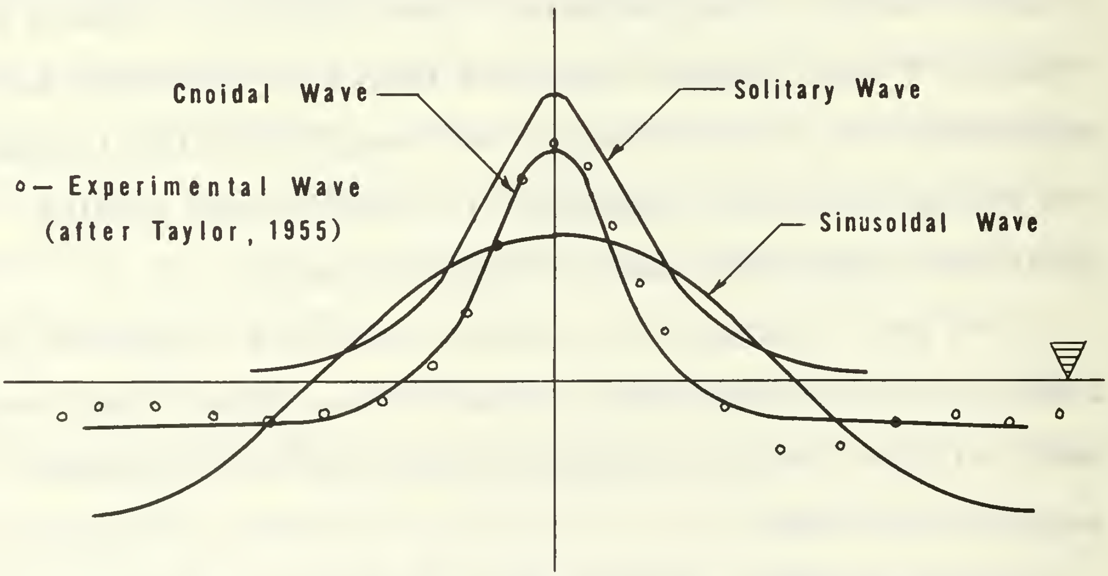

Figure 2

Experimental & Theoretical Wave Profiles

(after Wiegel, 1955)

approximated by hyperbolic functions and other assumptions are made which further simplify the theory. A more detailed discussion of hyperbolic wave theory follows.

III. DEVELOPMENT OF HYPERBOLIC WAVE EQUATIONS

Iwagaki [*op. cit.*] had defined a new class of waves derived from cnoidal theory and has termed them hyperbolic waves. The term hyperbolic comes from the fact that the Jacobian elliptic functions can be closely approximated by hyperbolic functions for certain restriced values of the argument.

Cnoidal theory is developed in terms of the Jacobian elliptic functions cn, sn, and dn; the elliptic modulus, k; and the complete elliptic integrals of the first and second kind, K and E, respectively. The theory is ordered on the ratio of wave height to water depth at the trough, $\frac{H}{h_t}$. By setting k and E equal to unity and allowing K to become infinite, the solitary wave can be formulated. In this case the wave length becomes infinite and the elliptic cosine and sine are replaced by sech and tanh, respectively.

Iwagaki has considered the case where k and E are set approximately equal to unity and K remains finite. From Figure 3, the relationships between k, K, and E can be determined. For the case where K = 3, both k and E are approximately equal to unity. With these considerations, cnoidal theory can be expressed in terms of hyperbolic functions and the complete elliptic integral of the first kind, K. From the above considerations, hyperbolic wave characteristics can be developed from cnoidal theory by putting

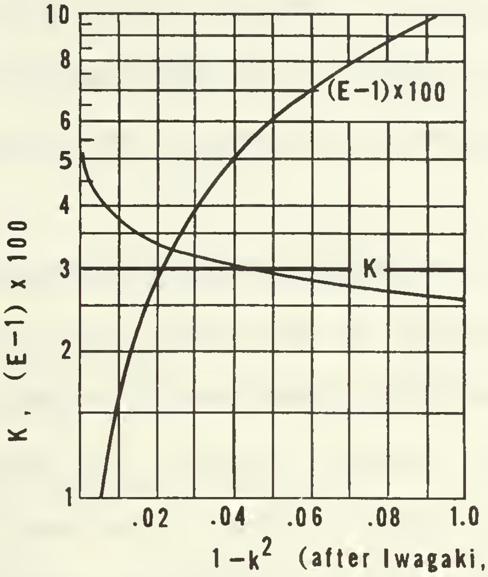

FIGURE 3

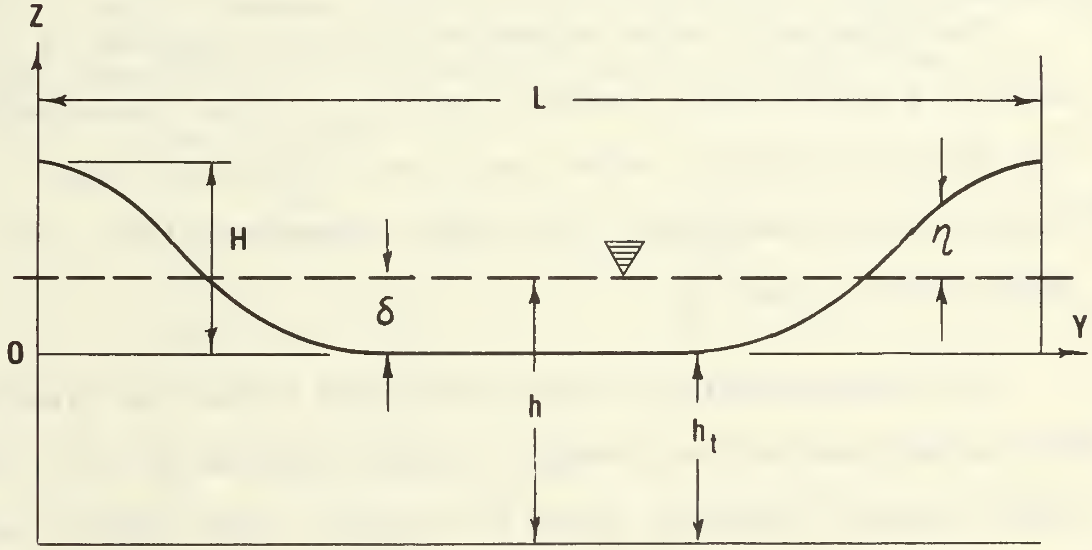

Figure 4

Wave Profile & Coordinate System

k and E equal to unity, keeping $K \geq 3$, and replacing the Jacobian elliptic functions by the appropriate hyperbolic functions. This limits the region of validity to very shallow water.

Traditionally cnoidal theory has used the depth of the water below the trough as the origin for the vertical axis. This same water depth is used with the wave height for ordering purposes. Iwagaki retains the same vertical axis origin, but by including a factor for the distance between the trough of the wave and the still water level, is able to order hyperbolic wave theory on the ratio of wave height to still water level. (Figure 4 shows the hyperbolic wave characteristics and axis.)

The hyperbolic wave approximations are applied to Laitone's second order cnoidal theory to obtain expressions for the wave profile, water particle velocities, wave celerity and wave energy. All wave characteristics are expressed to order $(\frac{H}{h})$.

Wave characteristics were developed using the hyperbolic wave approximation as given by Iwagaki applied to the first order cnoidal theory as given by Laitone. The results are listed below:

A. Wave Profile

$$\eta = H \operatorname{sech}^2 Q - \frac{H}{K} \tag{11}$$

$$\text{where } Q = \left(\frac{2K\ell_i x_i}{L} - \frac{2Kct}{L} \right) \tag{11a}$$

$$\text{and} \quad \ell_1 = \sin\alpha \qquad i = 1,2$$

$$\ell_2 = \cos\alpha$$

B. Horizontal Particle Velocity

$$u_i = \sqrt{gh}\left[\left(\frac{H}{h}\right)\operatorname{sech}^2 Q - \frac{1}{K}\left(\frac{H}{h}\right)\right]\ell_i \quad i = 1,2 \tag{12}$$

C. Vertical Particle Velocity

$$w = \sqrt{3gh}\left(1+\frac{z}{h}\right)\left(\frac{H}{h}\right)^{3/2} \operatorname{sech}^2 Q \tanh Q \tag{13}$$

(Note: For first order considerations, w is neglected. This is a result of the shallow water wave formulation, which assumes vertical water particle velocities are small compared to the horizontal particle velocities.)

D. Wave Celerity

$$C = \sqrt{gh}\left[1 + \frac{H}{h}\left(\frac{1}{2} - \frac{1}{K} - \frac{1}{2K}\right)\right] \tag{14}$$

E. Potential Energy

$$E_p = \frac{1}{3}\,\frac{\rho g H^2}{K}\left[1 - \frac{3}{2K}\right] \tag{15}$$

F. Kinetic Energy

$$E_K = \frac{1}{3}\,\frac{\rho g H^2}{K}\left[1 - \frac{3}{2K}\right] \tag{16}$$

G. Total Energy

$$E_t = E_p + E_K = \frac{2}{3} \frac{\rho g H^2}{K} [1 - \frac{3}{2K}] \tag{17}$$

There is an equal partitioning of potential and kinetic energy for the first order theory.

IV. RADIATION STRESS PHENOMENA

The concept of a "radiation stress" is a relatively recent one, having been developed in a series of papers by Longuet-Higgins and Stewart in the early 1960's. The original series of papers were rigorously developed, using detailed perturbation techniques. As the authors later admitted, the details of the mathematical development tended to obscure the relative simplicity of the concept based on physical reasoning. As a result, Longuet-Higgins and Stewart [10] have attempted to present a simplified, physical explanation of the radiation stress concept and resulting phenomena.

To illustrate the concept, the authors make an analogy to an occurance in electromagnetism. They note that when electromagnetic radiation originates or terminates on a surface, a force results and is known as a radiation pressure. In a like manner, an analogous force results from a wave train on a fluid surface. The resulting force is directed in the direction of wave propagation. Additionally, the wave train has momentum directed in the propagation direction. From the consideration of the conservation of momentum, any obstacle which impedes the wave train has a force exerted upon it and this force is equal to the rate of change of wave momentum. This force represents the radiation stress, which may be defined as an excess flow of momentum due to the wave train.

Longuet-Higgins and Stewart have used the radiation stress concept to explain numerous observed phenomena in the ocean [Refs. 11, 12, 13]. Among these are changes in the mean water level outside and inside the surf zone, known as the set-down and set-up, respectively. Other examples treated are surf beats, wave-current interaction and the generation of capillary waves by steep gravity waves. This investigation will consider the case of radiation stress applied to the set-down and set-up phenomena.

A. DEVELOPMENT OF RADIATION STRESS PHENOMENA

It is now appropriate to consider the development of phenomena resulting from the radiation stress. The case to be investigated considers waves from deep water which are progressing shoreward. As the waves shoal over a sloping bottom they increase in height until the breaking point is reached. As the wave height changes the radiation stress undergoes a corresponding change. The change in the radiation stress in turn causes changes in the mean water level.

After a wave initially breaks, it is assumed to continue across the surf zone with the wave height governed by the depth. This implies a spilling breaker classification. Implicit in the above assumption is a beach of gentle slope. One may then assume that no energy reflection will occur from the beach slope. Viscous considerations will be neglected. The net current perpendicular to the beach is assumed to be zero. The bottom is assumed to have straight and parallel

contours, although the bottom profile remains arbitrary. A final consideration for the formulation is a steady state system.

Consider the equation for the time-averaged conservation of momentum averaged over depth using Leibnitz' rule, as given by Phillips [13],

$$\frac{\partial \tilde{M}_i}{\partial t} + \frac{\partial}{\partial x_i}(\tilde{U}_i \tilde{M}_j + S_{ij}) = T_i + R_i \quad i,j = 1,2 \quad (19)$$

where (1,2) refer to the (x,y) components respectively.

The first term represents a change in the total horizontal momentum. The first term in the parenthesis represents a momentum flux due to steady motion and the second term is a momentum flux due to unsteady motion - the radiation stress tensor. The first term on the right side of (19) is a horizontal force due to the slope of the free surface. The last term is a friction term.

1. Development of the Radiation Stress Tensor

In developing the radiation stress tensor, the coordinate system represented in Figure 5 is utilized. The Y-axis is perpendicular to the beach, the X-axis is parallel to the beach, and the Z-axis is directed upward. The origin of the coordinate system is centered at the still water level. As a general case, the waves are assumed to approach the beach contours at an arbitrary angle, α. The development retains terms to second order only. For the shallow water case under consideration, terms are ordered on the ratio of the wave height to still water depth, $\frac{H}{h}$.

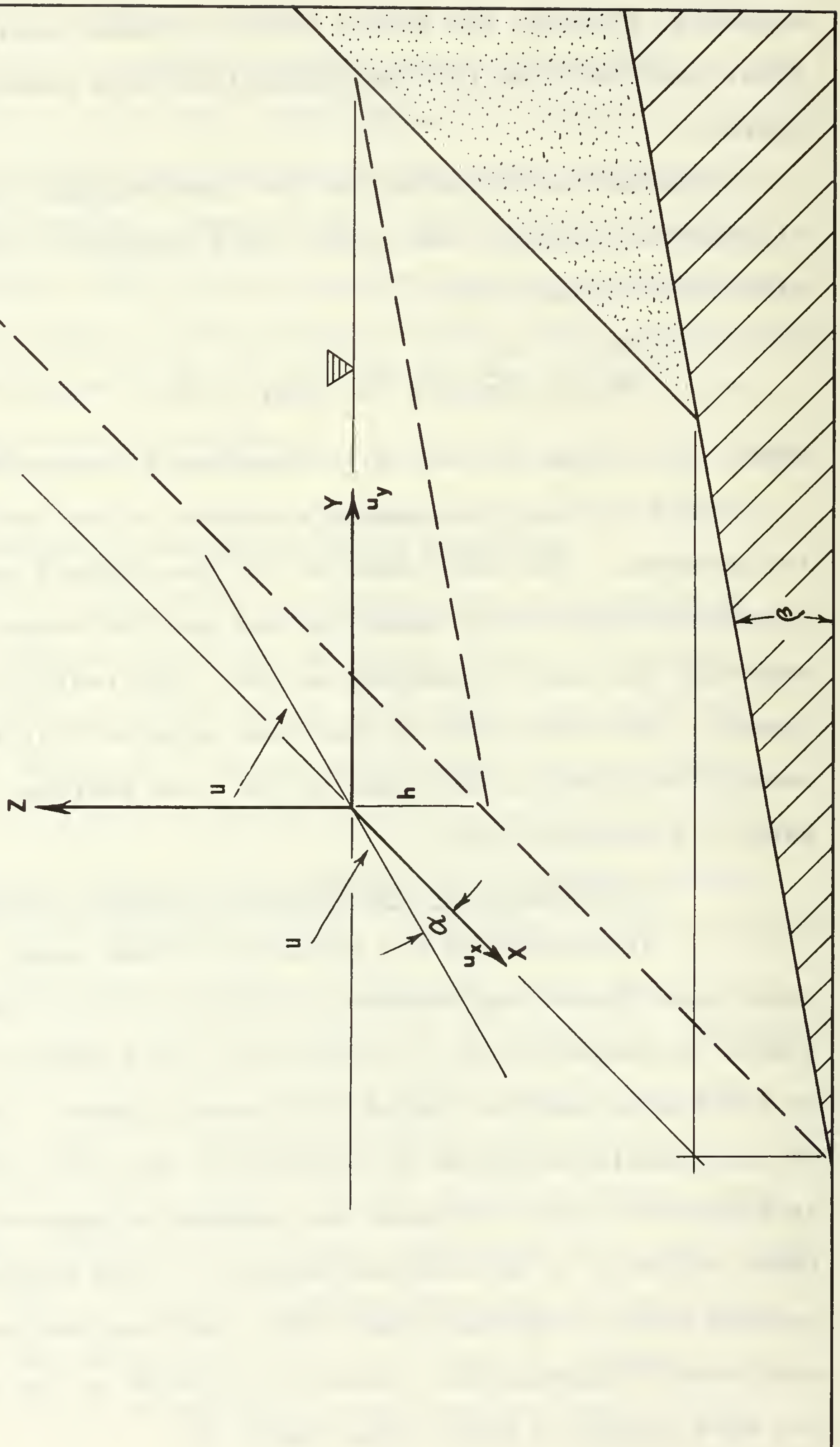

FIGURE 5: Coordinate System

Following the development of Longuet-Higgins and Stewart [10], consider the flux of horizontal momentum across a vertical plane in the fluid. With the assumption of no mean currents and considering only unsteady motion, this momentum flux can be represented by

$$\rho\ (u_x^{\ 2} + u_y^{\ 2}) + p \quad . \tag{20}$$

The total average flux of horizontal momentum across a vertical plane is found by integrating equation (20) between the bottom and the free surface and time averaging:

$$\overline{\int_{-h}^{\eta} (\rho u_y^{\ 2} + u_x^{\ 2} + p)\ dz} \tag{21}$$

where the over bar indicates time averaging.

By considering only components in the Y-direction, the principle component of the radiation stress, S_{yy}, is defined as the mean value of (21) minus the mean flux in the absence of the wave train:

$$S_{yy} = \overline{\int_{h}^{\eta} (\rho u_y^{\ 2} + p) dz} - \overline{\int_{-h}^{o} p_o\ dz} \tag{22}$$

where S_{yy} represents the excess momentum flux due to unsteady wave motion. The second term of (22) represents the hydrostatic contribution of the total pressure in the absence of any mean flow.

By separating (22) into three parts and integrating each separately, the problem is simplified. Consider

$$S_{yy} = S_{yy}(1) + S_{yy}(2) + S_{yy}(3) \tag{23}$$

where $S_{yy}(1) = \overline{\int_{-h}^{\eta} \rho u_y^{\,2} dz}$

$$S_{yy}(2) = \overline{\int_{-h}^{o} (p - p_o) dz} \tag{24}$$

$$S_{yy}(3) = \overline{\int_{o}^{\eta} p dz} \quad .$$

All terms of (24) contain terms of second order. Terms of higher order will be discarded.

Beginning with the $S_{yy}(1)$ term, the upper limit may be replaced by z = 0, since the portion from $0<z<\eta$ can be shown to give a third order contribution. Thus,

$$S_{yy}(1) = \overline{\int_{-h}^{o} \rho u_y^{\,2} dz} = \int_{-h}^{o} \overline{\rho u_y^{\,2}} dz \tag{25}$$

represents a Reynolds' stress integrated from the bottom to the free surface.

Considering the $S_{yy}(2)$ term,

$$S_{yy}(2) = \overline{\int_{-h}^{o} (p - p_o) dz} = \int_{-h}^{o} (\bar{p} - p_o) dz \tag{26}$$

which represents changes in the mean pressure within the fluid column. To be consistent $\bar{p}$ must be considered to the second order. Longuet-Higgins and Stewart note that $\bar{p}$ may

be easily derived from the vertical momentum flux equation across a horizontal plane. This flux must be great enough to support the weight of the water above. Then,

$$\overline{p + \rho w^2} = - \rho g z = p_o \tag{27}$$

$$\text{and} \quad \overline{p - p_o} = - \overline{\rho w^2} \quad . \tag{28}$$

Substituting (28) into (26) and combining with (25) gives

$$S_{yy}(1) + S_{yy}(2) = \int_{-h}^{o} \overline{\rho (u_y{}^2 + w^2) dz} \quad . \tag{29}$$

The $S_{yy}(3)$ term is evaluated by considering that the pressure near the free surface is nearly equal to the hydrostatic pressure below the free surface. Thus for

$$p = \rho g (\eta - z) \tag{30}$$

integrated over the stated limits and time averaged gives

$$S_{yy}(3) = \frac{1}{2} \rho g \bar{\eta}^2 \quad . \tag{31}$$

Combining all terms yields

$$S_{yy} = \int_{-h}^{o} \rho (\overline{u_y{}^2 + w^2}) dz + \frac{1}{2} \rho g \bar{\eta}^2 \tag{32}$$

2. <u>Wave Set-Down</u>

When the stated assumptions are applied to equation (19), it may be reduced to a simple expression. The first term in equation (19) is discarded from the steady state assumption. The first term in the parenthesis is neglected from the assumption of no net currents perpendicular to the

beach and the fact that there are no current gradients in the X-direction. The friction term is neglected. Equation (19) is now reduced to the following expression:

$$\frac{dS_{yy}}{dy} = T_y \quad . \tag{33}$$

The right side of (33) is a horizontal force resulting from the slope of the free surface and is given by

$$T_y = -\rho g(\bar{\eta} + h)\frac{d\bar{\eta}}{dy} \quad . \tag{34}$$

Making the assumption that $\eta \ll h$, (33) can be rearranged to give

$$\frac{d\bar{\eta}}{dy} = -\frac{1}{\rho g h}\frac{dS_{yy}}{dy} \tag{35}$$

which is the basic equation for considering the changes in mean level due to radiation stress.

For regions seaward of the point where the wave begins to break, Longuet-Higgins and Stewart [11] have presented a solution for the mean water level using linear waves given by

$$\bar{\eta} = -\frac{1}{8}\frac{H^2 m}{\sinh 2mh} \tag{36}$$

where the negative sign indicates a set-down. Equation (36) is expressed in terms of the local wave height, wave number and water depth. An additional expression is given in terms of the deep water wave height and wave number and is written

$$\bar{\eta} = -\frac{1}{8} H_o^2 m_o \frac{\coth^2 mh}{2mh + \sinh 2mh} \tag{37}$$

From equations (36) and (37) it is readily seen that the set-down increases as the wave shoals, as long as the energy flux remains constant. In addition, the assumption of an arbitrary bottom profile is valid, since the amount of set-down is only dependent on the local water depth, rather than any specified bottom profile.

3. Wave Set-Up

The set-down reaches its maximum value at wave breaking and other assumptions come into prominence. As the wave breaks and continues shoreward, energy is being continually released. The energy decrease across this region is assumed to be a function of the local water depth. Additionally, it is assumed that the S_{yy} terms retain the same form as (32). The energy decrease results in a corresponding decrease in the radiation stress and results in a set-up.

The solution for set-up using linear theory is given by

$$\bar{\eta} = \gamma\,(h_b - h) + \bar{\eta}_b \tag{38}$$

where
$$\gamma = \frac{1}{(\frac{8}{3}\,\kappa^2 + 1)} \tag{38a}$$

and
$$\kappa = \frac{H_b}{h_b} \;. \tag{38b}$$

Equation (38b) represents the value of the shoaling coefficient at breaking. The subscript, b, refers to conditions at breaking. The $\bar{\eta}_b$ from (38) is the maximum value of set-down, or the set-down at incipient breaking.

V. CNOIDAL WAVE CHARACTERISTICS APPLIED TO RADIATION STRESS EQUATIONS

This section discusses the application of the cnoidal wave characteristics developed earlier to the radiation stress equations. As a first step, the radiation stress term will be developed. This will then be used to develop the expressions for set-down and set-up.

A. THE RADIATION STRESS TENSOR

Because only the primary term of the radiation stress tensor, S_{yy}, is required for the simple case of waves approaching perpendicular to the bottom contours, only cursory consideration will be given to the remaining three terms of the tensor. For other investigations requiring a more general solution, the remaining terms of the tensor are easily developed from this simple case. Once again, only terms of second order are required for the present consideration.

Equation (32) specified the complete form of the S_{yy} term, which was then separated into three components for ease in development. A similar approach will be used for the application of the cnoidal wave characteristics. The $S_{yy}(1)$ and $S_{yy}(2)$ terms were combined to give

$$S_{yy}(1) + S_{yy}(2) = \int_{-h}^{0} \overline{\rho(u_y^{\,2} + w^2)dz}$$

which is equation (29).

The component of horizontal particle velocity in the Y-direction (12) can now be written as

$$u_y = \sqrt{gh}\ [(\frac{H}{h})\mathrm{sech}^2 Q - \frac{1}{k}\ (\frac{H}{h})]\cos\alpha \tag{39}$$

Squaring (49) gives

$$u_y^{\ 2} = gh\ (\frac{H}{h})^2\ \cos^2\alpha\ [\mathrm{sech}^4 Q - \frac{2}{k}\ \mathrm{sech}^2 Q + \frac{1}{k^2}] \tag{40}$$

Upon consideration of the equation for the vertical particle velocity (13), it is noted that when this term is squared it contains terms of order $(\frac{H}{h})^{5/2}$. Since only terms of order $(\frac{H}{h})^2$ are to be retained, equation (32) can now be written as

$$S_{yy}(1) + S_{yy}(2) = \int_{-h}^{0} \overline{\rho u_y^{\ 2}} dz \tag{41}$$

Substituting equation (40) into (41), integrating over the prescribed limits and averaging over one wave length results in the following expression:

$$S_{yy}(1) + S_{yy}(2) = \frac{2}{3}\ \frac{\rho g H^2}{K}\ [1 - \frac{3}{2K}]\ \cos^2\alpha \tag{42}$$

For the case of waves approaching perpendicular to the bottom contours, equation (42) is identical to the expression for total energy per unit surface area (17).

The $S_{yy}(3)$ term, equation (31) was written

$$S_{yy}(3) = \frac{1}{2}\ \rho g \bar{\eta}^2$$

and is equivalent to the mean potential energy per unit surface area. Squaring the expression for the wave profile (11) gives

$$\eta^2 = H^2 \left(\operatorname{sech}^4 Q - \frac{1}{K^2}\right) \quad . \tag{43}$$

Averaging (43) over one wave length and substituting into (31) results in

$$S_{yy}(3) = \frac{1}{3}\frac{\rho g H^2}{K}\left(1 - \frac{3}{2K}\right) \tag{44}$$

Equation (44) is the same as the expression for the mean potential energy per unit surface area.

Combining (42) and (44) gives

$$S_{yy} = E_t \cos^2\alpha + \frac{E_t}{2} \tag{45}$$

to the second order.

In a similar manner, the remaining terms of the radiation stress tensor can be developed and the tensor can be written

$$S_{ij} = \begin{vmatrix} E_t \sin^2\alpha + \frac{E_t}{2} & \frac{E_t}{2}\sin 2\alpha \\ \frac{E_t}{2}\sin 2\alpha & E_t\cos^2\alpha + \frac{E_t}{2} \end{vmatrix} \quad .$$

This is the same form in terms of energy density as the tensor for the shallow water linear wave development.

Considering the case where the waves approach perpendicularly to the beach ($\alpha = 0$), (45) may now be written

$$S_{yy} = \frac{3}{2} E_t \quad . \tag{46}$$

Equation (56) is of the same form as the expression developed earlier for shallow water gravity waves. This is considered to be a significant result, since it now allows an analogous development of the set-down and set-up expressions for cnoidal theory. Additionally, it should provide a basis for comparison of the first order cnoidal and linear theory radiation stress results.

B. CNOIDAL WAVE SOLUTION FOR SET-DOWN

The general form of the equation for set-down is given by (35). An analytical solution for the hyperbolic approximation to cnoidal theory will be developed for the limiting case of K = 3.

Recall that equation (35) was expressed approximately outside the surf zone as

$$\frac{d\bar{\eta}}{dy} = - \frac{1}{\rho g h} \frac{dS_{yy}}{dy}$$

where $h = h(y)$ and $S_{yy} = \frac{3}{2} E_t$ for $\alpha = 0$. For the region seaward of the point where initial breaking occurs, the mean rate of energy transport is constant and is written as

$$F = E_t C_g \quad . \tag{47}$$

For the shallow water consideration $C_g = C$. Longuet-Higgins has presented a general solution for equation (35) as shown below:

$$\bar{\eta} = \frac{\sigma^3 F}{\rho g^3} \frac{d}{d\theta} \left(\frac{\xi}{\theta}\right) + \text{const.} \tag{48}$$

In equation (58), σ is the radial frequency, F is the mean rate of energy transport. ξ and θ are dimensionless

parameters as defined below for the case where K = 3.

$$\xi = mh = \frac{4Kh}{L} = \sqrt{3}\left(\frac{H}{h}\right)^{1/2} \frac{1}{\left(1-\frac{1}{2}\frac{H}{h}\right)} \tag{49}$$

In equation (49) $\frac{4K}{L}$ is analogous to the wave number of linear wave theory.

$$\theta = \frac{\sigma^2 h}{g} = 3\left(\frac{H}{h}\right) \frac{1}{\left(1-\frac{1}{2}\frac{H}{h}\right)^2} \quad . \tag{50}$$

From (49) and (50) it can be shown that

$$\theta = \xi^2 \quad .$$

Substituting (49) and (50) into (48) gives:

$$\bar{\eta} = \frac{\sigma^3 F}{\rho g^3} \frac{d}{d\theta} \left(\frac{1}{\theta^{1/2}} \right) \quad . \tag{51}$$

Integrating (51) with respect to θ and substituting into the result gives:

$$\bar{\eta} = -\frac{1}{2} \frac{\sigma^3 F}{\rho g^3} \left(\frac{g}{\sigma^2 h}\right)^{3/2} = -\frac{1}{2} \frac{F}{\rho (gh)^{3/2}} \tag{52}$$

For the limiting case of K = 3, the celerity reduces to

$$C = \sqrt{gh}$$

and substituting $E = \frac{\rho g H^2}{9}$ into equation (52), the change in mean water level outside the breaking point is:

$$\bar{\eta} = -\frac{1}{18} \frac{H^2}{h} \tag{53}$$

For the cases where K > 3, equation (35) does not lend itself to an analytical solution and a numerical scheme must be devised for the solution. Using equation (47) for the constancy of energy transport outside the breaking point, a

wave height can be calculated for each depth. A numerical difference scheme is then applied to equation (35) to give a solution for each point as specified by a given water depth.

C. CNOIDAL SOLUTION FOR SET-UP

The general solution for the set-up after the inception of breaking is obtained by considering once again equation (35) and the same assumption from the linear theory case. Inside the surf zone a solution for the general case is readily obtained by analytical means. The primary tensor term (46) can be written:

$$S_{yy} = \frac{3}{2} E_t = \frac{\rho g H^2}{K} \left(1-\frac{3}{2K}\right) \tag{54}$$

Inside the point of initial breaking, the total mean water depth is now represented by

$$D = (\bar{\eta} + h) \tag{55}$$

in which $h = h(y)$ and $\bar{\eta}$ is the set-up. The critical value of the shoaling coefficient now becomes

$$\kappa = \frac{H_b}{D_b} \quad . \tag{56}$$

Thus, equation (54) becomes

$$S_{yy} = \frac{\rho g \kappa^2{}_b D(y)^2}{K} \left(1-\frac{3}{2K}\right) \quad . \tag{57}$$

Substituting (57) into (35) gives:

$$\frac{d\bar{\eta}}{dy} = -\frac{2\kappa^2{}_b}{K} \left(1-\frac{3}{2K}\right) \frac{dD}{dy} \tag{58}$$

or

$$\frac{d\bar{\eta}}{dy} = - \frac{2\kappa^2 b}{K} \left(1-\frac{3}{2K}\right) \left(\frac{d\bar{\eta}}{dy} + \frac{dh}{dy}\right) \qquad (59)$$

Now let

$$\frac{1}{A} = \frac{2\kappa^2 b}{K} \left(1-\frac{3}{2K}\right) \qquad (60)$$

Rewriting (59)

$$\frac{d\bar{\eta}}{dy} = - B \frac{dh}{dy} \qquad (61)$$

where $B = \frac{1}{(A+1)}$

Equation (61) can now be integrated to give:

$$\bar{\eta} = - Bh(y) + \text{const.} \qquad (62)$$

At initial breaking $\bar{\eta} = \bar{\eta}_b$

and $h = h_b$.

Thus the constant in (62) becomes

$$\text{const.} = \bar{\eta}_b + Bh_b \quad . \qquad (63)$$

Substituting (63) into (62) gives the general solution for set-up inside the surf zone as

$$\bar{\eta} = B(h_b - h) + \bar{\eta}_b \qquad (64)$$

where $h = h(y)$.

Equation (64) has the same form as the linear solution.

VI. APPLICATION OF THEORY TO EXPERIMENTAL DATA

Bowen, *et al*. [14] conducted a laboratory experiment to carefully measure the set-down and set-up caused by shoaling waves. The experiment was conducted in a laboratory channel with a plane beach of slope 1:12. The measurements were made using a sensitive manometer arrangement, whose pressure readings were later converted to water surface elevations.

Seaward of the inception of breaking, a steadily increasing value of set-down was measured. As the wave began to break, a near constant value of set-down was measured, until breaking was complete. This suggests a wave more nearly plunging, rather than the spilling breaker considered in the theoretical developments. Once initial breaking was complete, a rebound phenomenon was noted where the broken wave formed a bore and proceeded shoreward. In this region, the set-up was found to steadily increase and the wave height was nearly a linear function of water depth. This is one of the considerations made in the theoretical developments.

A particular laboratory measurement was chosen to compare linear theory and cnoidal theory. The set-down and set-up calculations were made for both cnoidal theory and the shallow water approximation to linear theory.

For the shallow water approximation to linear theory, equation (36) becomes

$$\bar{\eta} = -\frac{1}{16}\frac{H^2}{h} . \qquad (65)$$

From the earlier development for the limiting case, equation (53), gives a comparable expression for cnoidal theory of

$$\bar{\eta} = - \frac{1}{18} \frac{H^2}{h} \quad .$$

Equations (65) and (53) were used to calculate the values of set-down. Equations (38) and (64) were used to calculate the values of set-up. A comparative plot for both theories and the measured data is shown in Figure 6.

For the theoretical solutions, it is necessary to patch the solutions for set-down and set-up together at the initial break point, since neither theory accounts for the transition zone indicated by the laboratory experiment. Also neither theory makes an allowance for the formation of the observed bore.

For K = 3 cnoidal theory gives a value of set-down and the corresponding set-up which is closer to the laboratory experiment than linear theory. When the value of K is increased, the resulting set-down decreases. For K = 6 the resulting cnoidal theory set-down gives a very good approximation to the laboratory experiment, particularly at the deeper measurement points where friction effects are likely to be minimized. This is an important point to note, since friction considerations were neglected in both theoretical developments. In both cases the assumption of a constant mean rate of energy transmission was made for the region outside the initial break point. By neglecting friction effects, both

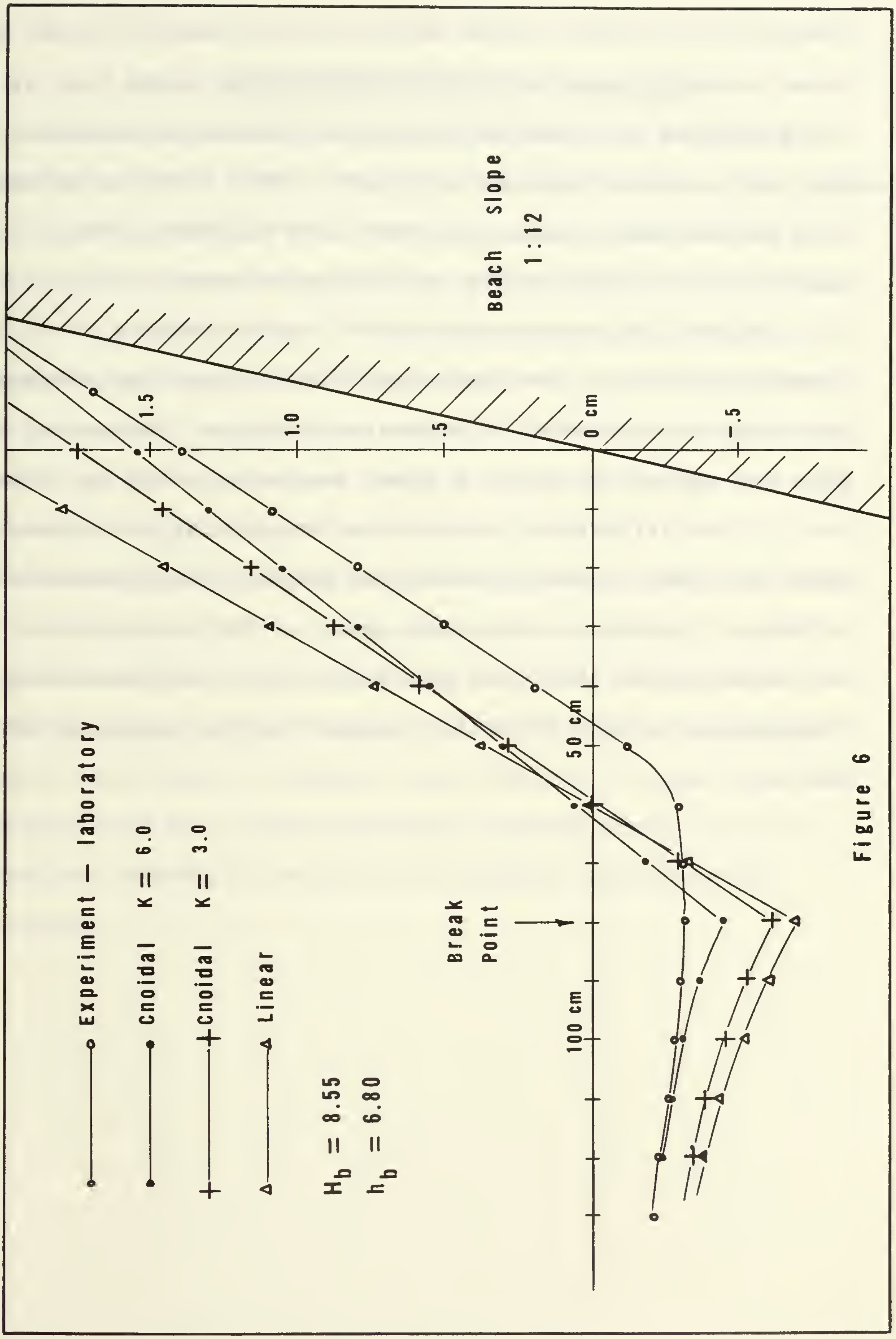
Beach Slope
1:12
Experiment — laboratory
Cnoidal K = 6.0
Cnoidal K = 3.0
Linear
$H_b = 8.55$
$h_b = 6.80$
Break Point
1.5
1.0
.5
0 cm
-.5
50 cm
100 cm

Figure 6

theories give energy values which are necessarily larger than those actually observed. It should also be noted that as the value of K is increased in the cnoidal theory expressions, the energy value decreases and gives lower set-down values. As K becomes very large, the wave form approaches the solitary wave condition for which the set-down is zero.

Because the set-up solution for both theories is a linear function of the decreasing water depth, the excessive set-down values result in excessive values of the set-up. Even the optimum value of K gives too large a set-up value. Some of the difference between the theoretical and laboratory results is due to the necessity of patching the theoretical solutions together at the break point. The theoretical set-up begins at the break point, whereas, the laboratory experiment showed a transition region before the onset of the set-up.

VII. RECOMMENDATIONS FOR FURTHER STUDY

Because this study is limited in scope, a number of suggestions for further investigations can be made. The simple relationships derived from the hyperbolic approximation to cnoidal theory lend themselves to other applications of the radiation stress concept. Extensions into the derivation of longshore currents and sediment transport by these currents could be made. Applications to surf beats and the effects of the set-up on storm surges could be done, following the work of Longuet-Higgins and Stewart.

The hyperbolic approximation to cnoidal theory has shown a useful application to radiation stress concepts. A far wider and more accurate study could be done using cnoidal theory as originally developed, rather than the asymptotic form of the theory. However, this would be a great deal more difficult and most certainly would require facility with numerical methods to utilize the elliptic integrals and functions.

VIII. SUMMARY AND CONCLUSIONS

The hyperbolic approximation to cnoidal wave theory can be readily applied to the radiation stress phenomena of set-down and set-up. Using techniques established for linear wave theory, wave characteristics for the cnoidal approximation yield an analytical solution for the limiting case of $K = 3$. For any case of K greater than three, a numerical scheme is required for solution. The same assumptions are made for both the linear and cnoidal models. The radiation stress tensor derived for first order cnoidal theory has the same form in terms of energy density as the tensor for linear theory.

The set-down solution for linear theory and cnoidal theory is a function of wave height and local water depth. In addition, cnoidal theory has a dependence upon the particular value of K under consideration. The values of set-down obtained for linear theory and the limiting cnoidal case ($K = 3$) were greater than the measured laboratory value. The limiting cnoidal case was closer to the measured values than linear theory. An optimum value of $K = 6$ was found for the particular experiment being considered. The resulting cnoidal values were in very close agreement with the laboratory values for the deeper measurement stations. The differences between the measured values of set-down and the theoretical values are attributed in part to the simplifying assumptions made in developing the mathematical models. For example, all friction

considerations were neglected. This assumption probably makes a considerable contribution to the excessive set-down resulting from both theories, particularly at the shallower stations.

At the point of initial breaking, the set-down and set-up solutions must be patched together. The excessive set-down values obtained from theory lead to larger than measured set-up results. The set-up solution is a linear function of water depth and the set-up continues above still water level on the beach. No provision is made in either model for the bore which was observed in the laboratory study.

Cnoidal theory appears to give a finite wave solution which more closely represents the observed results from the laboratory experiment. As the value of K increases, the resulting set-down and set-up approach the laboratory results. At K = 6, good agreement between cnoidal theory and the measured values is found, especially in deeper water. Thus cnoidal theory is a more valid theory in shallow water, not only from mathematical considerations, but from physical measurements as well.

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13. ABSTRACT

A simplified hyperbolic approximation to cnoidal wave theory is applied to generate the radiation stress tensor and used in the equations of motion to obtain solutions for changes in the mean water level outside and inside the surf zone. Comparison between linear theory, cnoidal theory and laboratory results are made. A limiting case and an optimum case for cnoidal theory are discussed in the comparison. Cnoidal theory is shown to give better predictions than linear theory to the data considered.

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